Teaching and Learning Calculus using ultrasmall numbers

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Continuity

$$(\forall \varepsilon > 0)\ (\exists \delta > 0)\ (\forall x) \quad |x - a| \leq \delta \Rightarrow |f(x) - f(a)| \leq \varepsilon$$

For too complicated to be used at introductory pre-university level.

Choice (in high school):

- lose rigour
- change paradigm $\longleftarrow$ why we are together

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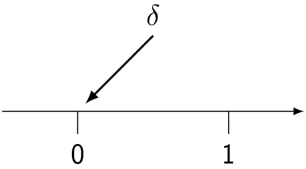

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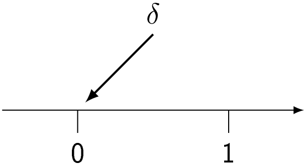

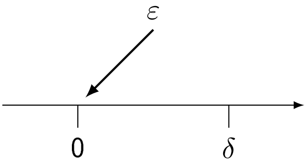

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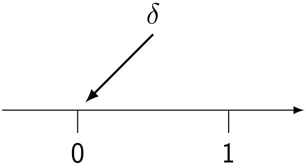

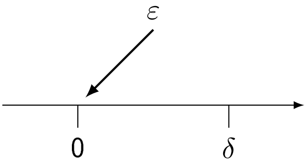

Formalisation:

Axioms: standard observability

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$$0,\ 1,\ \sqrt{2},\ e,\ \pi$$

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(4) Every real number x which is not ultralarge can be written as $x = a + \delta$, where a is standard and δ is ultrasmall or zero. We write

$$x \simeq a$$

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Axioms: generalisation

Definition (context)

The context of a statement, function or set, is the list of parameters used in its definition.
(A context may be empty)

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Standard numbers are observable relative to the empty context, or relative to every context.
Every number is observable in some context.

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Implies: Operations between observable real numbers yield observable results.

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Do not worry about the context provided it contains at least all parameters involved.

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- Most theorems are easier to understand, easier and shorter to prove (no weird behaviour "at the limit")
- But: the one about Cauchy sequences is so simple, something must be wrong...

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A sequence $\{a_n\}$ converges to a limit if there is an observable value L such that for any ultralarge index N, we have

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Theorem

A sequence converges to a limit iff it is a Cauchy sequence.

Proof

$\Rightarrow$ (easy)
for any ultralarge indexes N and M,

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Assume N is a fixed ultralarge index, and M is any ultralarge index.
Then $a_N \simeq a_M$,
so there is a value ℓ such that for any ultralarge index M, we have $a_M \simeq \ell$.

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By closure there is an observable such value and this is the limit L. $\square$

Lucien's comment

This seems to use another "nature" of real numbers.

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Metaphysical question?

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2. but we all know that infinitesimals don't really exist
3. I will believe you when you can show me an infinitesimal
4. we don't want analysis to be taught that way

Time to cease to be polite

Foundational incompetence and religious belief in a metaphysical truth about real numbers are not acceptable in a scientific discussion.

Valid questions

1. Is it mathematically useful?
2. Is it pedagogically useful?

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   Consider

$$H : x \mapsto \frac{1}{2} + \frac{1}{\pi} \arctan\left(\frac{x}{\varepsilon}\right) \quad \text{for ultrasmall positive } \varepsilon$$

   At standard scale: Heaviside function. Its derivative – at standard scale – is a Dirac. *And they can be multiplied!*[2]

[2] the product is a lopsided Dirac...

2. Is it pedagogically useful?

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Final exams have an external jury, from university.
Jury was Prof. Laura Weiss, professor of didactics of mathematics and physics who was also jury for other teachers.

Continuity of $y = x^2$

$$\text{Is } f : x \mapsto x^2 \text{ continuous at } x = 3?$$

Ultracalculus: yes because, for $h \simeq 0$ we have
$(3 + h)^2 = 9 + 6h + h^2$

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$(3 + h)^2 = 9 + 6h + h^2$
$6h \simeq 0$ because 6 is observable and h is ultrasmall, and observable times ultrasmall is ultrasmall (proof available).
Same for h^2:
hence $9 + 6h + h^2 \simeq 9 = 3^2$.

classical approach

Yes, because

$$\lim_{x \to 3} x^2 = 3^2$$

(with no extra comment...)

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or

$$\lim_{h \to 0} (3+h)^2 = 3^2 + 6h + h^2 = 3^2 + \lim_{h \to 0} 6h + \lim_{h \to 0} h^2 = 3^2$$

notice the absence of "lim" at second step.

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"The lim *notation seems used as a prefix, with no clear meaning."*

continuity of the product

Ultracalculus

$$x \simeq a \Rightarrow x = a + dx$$

$$f(a + dx) - f(a) = \Delta f(a) \simeq 0 \text{ if } f \text{ is continuous at } a$$

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$$f(a+dx) \cdot g(a+dx) = (f(a) + \Delta f(a)) \cdot (g(a) + \Delta g(a))$$

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$$f(a+dx)\cdot g(a+dx) = (f(a)+\Delta f(a))\cdot(g(a)+\Delta g(a))$$

$$= f(a)\cdot g(a) + \underbrace{f(a)\cdot\Delta g(a)}_{\simeq 0} + \underbrace{\Delta f(a)\cdot g(a)}_{\simeq 0} + \underbrace{\Delta f(a)\cdot\Delta g(a)}_{\simeq 0}$$

$$\simeq f(a)\cdot g(a)$$

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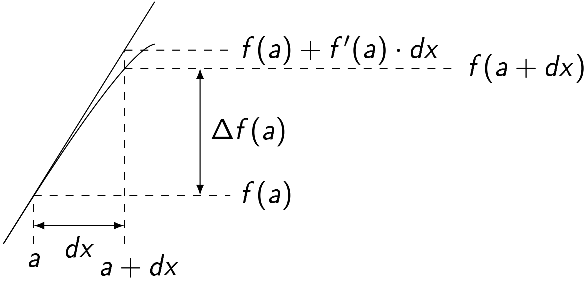
It is another problem so it has another rule.

variable substitution

Ultracalculus



$df(a) = f'(a) \cdot dx$ (the variation along the tangent)

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then for $\sqrt{1+2x} = u$ we have

$$\frac{du}{dx} = \frac{1}{\sqrt{1+2x}} = \frac{1}{u} \text{ hence } u \cdot du = dx$$

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All together

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$$\int_1^{\sqrt{3}} u^2 \cdot du$$

a step by step method

classical approach

Need to prove, using the fundamental theorem twice, that

$$\int_a^b f(x)dx = \int_{\varphi^{-1}(a)}^{\varphi^{-1}(b)} f(\varphi(t))\varphi'(t)dt$$

and use it in one step.

Laura Weiss's comment

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- In the classical approach, the technicality is either so high that there is no energy left for understanding, or everything is hidden under the carpet, but then proofs are mere metaphors.
- Students working with ultracalculus have better and safer tools.
- Because they can rely on algebra, when they are lost or in error, students working with ultracalculus can use a hint much more efficiently.

Laura Weiss's comment

"Descartes made geometry algebraic and that was a great improvement.
Analysis with ultrasmall numbers make analysis more algebraic and it seems to provide the same sort of advantage."

Thank you